

# On localization for the Schrödinger operator with a Poisson random potential

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## Abstract

We prove exponential localization for the Schrödinger operator with a Poisson random potential at the bottom of the spectrum in any dimension. We also prove exponential localization in a prescribed interval for all large Poisson densities. In addition, we obtain dynamical localization and finite multiplicity of the eigenvalues.

## Résumé

On démontre localisation exponentielle pour l'opérateur de Schrödinger avec un potentiel aléatoire de Poisson, pour les basses energies et en toute dimension. On démontre aussi localisation exponentielle dans un intervalle d'énergies donné et à grande densité. On obtient de plus localisation dynamique et le fait que la multiplicité des valeurs propres est finie.

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## 1. Results

The Poisson Hamiltonian is the random Schrödinger operator on  $L^2(\mathbb{R}^d)$  given by

$$H_X = -\Delta + V_X, \quad \text{with} \quad V_X(x) = \sum_{\zeta \in X} u(x - \zeta), \quad (1)$$

where the single-site potential  $u$  is a nonnegative  $C^1$  function on  $\mathbb{R}^d$  with compact support—without loss of generality we take  $u(x) \leq u(0) = 1$ —and  $V_X$  is a Poisson random potential, that is,  $X$  is a Poisson process on  $\mathbb{R}^d$  with density  $\varrho > 0$ . Thus the configuration  $X$  is a random countable subset of  $\mathbb{R}^d$ , and,

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letting  $N_X(A)$  denote the number of points of  $X$  in the Borel set  $A \subset \mathbb{R}^d$ , each  $N_X(A)$  is a Poisson random variable with mean  $\varrho|A|$  (i.e.,  $\mathbb{P}_\varrho\{N_X(A) = k\} = (\varrho|A|)^k (k!)^{-1} e^{-\varrho|A|}$  for  $k = 0, 1, 2, \dots$ ), and the random variables  $\{N_X(A_j)\}_{j=1}^n$  are independent for disjoint Borel sets  $\{A_j\}_{j=1}^n$ . We will denote by  $(\mathcal{X}, \mathbb{P}_\varrho)$  the underlying probability space for the Poisson process with density  $\varrho$ .

Note that  $H_X$  is an ergodic (with respect to translations in  $\mathbb{R}^d$ ) random self-adjoint operator. It follows that the spectrum of  $H_X$  is the same for  $\mathbb{P}_\varrho$ -a.e.  $X$ , as well as the decomposition of the spectrum into pure point, absolutely continuous, and singular continuous spectra. For  $u$  as above we actually get  $\sigma(H_X) = [0, +\infty[$  for  $\mathbb{P}_\varrho$ -a.e.  $X$  [9].

We prove exponential localization for Poisson Hamiltonians at the bottom of the spectrum. By  $\chi_B$  we denote the characteristic function of the set  $B \subset \mathbb{R}^d$ , with  $\chi_x$  denoting the characteristic function of the cube of side 1 centered at  $x \in \mathbb{R}^d$ . We write  $\langle x \rangle = \sqrt{1 + |x|^2}$ ,  $T(x) = \langle x \rangle^\nu$  for some fixed  $\nu > \frac{d}{2}$ .

**Theorem 1** *Given  $\varrho > 0$ , there exists  $E_0 = E_0(\varrho) > 0$  and  $m = m(\varrho) > 0$ , such that for  $\mathbb{P}_\varrho$ -a.e.  $X$  the following holds : the operator  $H_X$  has pure point spectrum in  $[0, E_0]$  with exponentially localized eigenfunctions with rate of decay  $m$ , i.e., if  $\phi$  is an eigenfunction of  $H_X$  with eigenvalue  $E \in [0, E_0]$ , there is a constant  $C_\phi < \infty$  such that*

$$\|\chi_x \phi\| \leq C_\phi e^{-m|x|} \quad \text{for all } x \in \mathbb{R}^d. \quad (2)$$

Moreover there exist constants  $\tau > 1$ ,  $s \in ]0, 1[$ , and  $C < \infty$ , such that for eigenfunctions  $\psi, \phi$  (possibly equal) with eigenvalue  $E \in [0, E_0]$  we have

$$\|\chi_x \psi\| \|\chi_y \phi\| \leq C \|T^{-1} \psi\| \|T^{-1} \phi\| e^{\langle y \rangle^\tau} e^{-|x-y|^s} \quad \text{for all } x, y \in \mathbb{Z}^d. \quad (3)$$

In particular, the eigenvalues of  $H_X$  in  $[0, E_0]$  have finite multiplicity, and  $H_X$  exhibits dynamical localization in  $[0, E_0]$ , that is, for any  $p > 0$  we have

$$\sup_t \|\langle x \rangle^p e^{-itH_X} \chi_{[0, E_0]}(H_X) \chi_0\|_2^2 < \infty. \quad (4)$$

For Poisson random potentials the density  $\varrho$  is a measure of the amount of disorder in the medium. The next theorem gives localization at high disorder.

**Theorem 2** *Given  $E_0 > 0$ , there exists  $\varrho_0 > 0$  such that for  $\varrho > \varrho_0$  the conclusions of Theorem 1 hold in the interval  $[0, E_0]$ .*

While Poisson Hamiltonians are the most natural random Schrödinger operators in the continuum (the distribution of impurities in a material being naturally modeled by a Poisson process), a mathematical proof of the existence of localization has been a long-standing open problem. Localization has been known only in one dimension [11]. A Poissonian model, which incorporates random intensities with bounded densities and requires single-site potentials that do not decay too slowly at infinity, was considered in [3].

In the multi-dimensional case, localization in the continuum had been proved for Anderson-type Hamiltonians with random intensities with bounded densities, e.g. [3], and for an  $\mathbb{R}^d$ -ergodic Schrödinger operator with a Gaussian random potential [5]; in both cases there is an “a priori” Wegner estimate obtained by averaging with bounded densities. But recently Bourgain and Kenig [2] proved localization for the Bernoulli-Anderson Hamiltonian, with the Wegner estimate being proven in a multiscale analysis.

To prove Theorems 1 and 2 we exploit the new ideas introduced by Bourgain and Kenig [1,2]. In particular, the control of the resonances (the Wegner estimate) is achieved by a multiscale analysis using “free sites” and a new quantitative version of unique continuation which gives a lower bound on eigenfunctions.

The control on the eigenfunction correlations given in (3) was introduced in [8]. That (3) implies dynamical localization is rather immediate. As for the finite multiplicity property, it follows by estimating  $\|\chi_x \chi_{\{E\}}(H_X)\|_2^2 \|\chi_y \chi_{\{E\}}(H_X)\|_2^2$  from (3) and summing over  $x \in \mathbb{Z}^d$ .

In the next section we outline the main ideas in the proof of Theorem 1; detailed arguments will be given in [6]. Theorem 2 is proved in a similar way, although the proof requires some modifications.

## 2. The main ideas

Given a cube  $\Lambda$  in  $\mathbb{R}^d$ , we let  $H_{X,\Lambda} = -\Delta_\Lambda + V_{X,\Lambda}$  be the restriction of  $H_X$  to  $\Lambda$  with Dirichlet boundary condition. We consider the finite volume resolvent  $R_\Lambda(E) = (H_\Lambda - E)^{-1}$  (we will omit  $X$  from the notation). By  $c, c', \dots$  we denote positive constants (not necessarily the same) independent of  $\varrho, L, \dots$ .

The multiscale analysis requires an *a priori* probabilistic estimate on the resolvent  $R_\Lambda(E)$  for all  $E \in [0, E_0]$  at a fixed, but sufficiently large, initial scale  $L_0$ , where  $\Lambda$  is a cube of side  $L_0$  centered at, say,  $x_0$ . To obtain this initial estimate for Theorem 1, we divide the cube  $\Lambda$  into non-overlapping cubes  $\Lambda(j)$  of side  $\ell_0 \approx (\varrho^{-1} \log L_0)^{\frac{1}{d}}$  centered at points  $j \in x_0 + \ell_0 \mathbb{Z}^d$ . We consider configurations such that  $1 \leq N(\Lambda(j)) \lesssim \varrho \ell_0^d$  for all  $\Lambda(j)$ , an event with high probability, more precisely, with probability  $\geq 1 - \left(\frac{L_0}{\ell_0}\right)^d L_0^{-p} \geq 1 - \varrho L_0^{-p+d}$ , where we can arrange for  $p$  large as desired.

For such configurations, we pick one  $\zeta_j \in \Lambda(j)$ , and split the potential as  $V_\Lambda = V_\Lambda^{(1)} + V_\Lambda^{(2)}$ , with  $V_\Lambda^{(1)} = \sum_j' u(x - \zeta_j)$ , where  $\sum_j'$  denotes the sum over sites  $j \in x_0 + 2\ell_0 \mathbb{Z}^d$  only; as a consequence the  $u(x - \zeta_j)$  in the sum are non-overlapping. We have  $0 \leq V_\Lambda^{(1)} \leq 1$  and  $V_\Lambda^{(2)} \geq 0$ . In order to estimate  $\|R_\Lambda(E)\|$ , it is convenient to use the operator  $\Gamma_\Lambda(E)$ , defined by

$$\Gamma_\Lambda(E) = (H_\Lambda^{(2)} + 1)^{-\frac{1}{2}}(1 + E - V_\Lambda^{(1)})(H_\Lambda^{(2)} + 1)^{-\frac{1}{2}}, \quad \text{with } H_\Lambda^{(2)} = -\Delta_\Lambda + V_\Lambda^{(2)} \geq 0. \quad (5)$$

Proceeding as in [2, Section 4], suppose  $\|\Gamma_\Lambda(E)\| > 1 - E_0$  with  $E_0$  small. Then there is  $g \in L^2(\mathbb{R}^d)$ , with  $1 - \sqrt{E_0} \leq \|g\| \leq 1$  and, using  $V_\Lambda^{(2)} \geq 0$ ,  $\|\nabla g\| \leq 2E_0^{\frac{1}{4}}$ , such that for each  $a \in \Lambda$  we have

$$0 \leq \langle \tau_a V_\Lambda^{(1)} g, g \rangle \leq cE_0^{\frac{1}{4}}(|a| + 1), \quad (6)$$

where  $\tau_a$  denotes translation by  $a$  and the estimate is uniform in  $L$  (cf. [2, Eqs. (4.7), (4.8), and (4.10)]). On the other hand, taking  $K = 10\ell_0$ , and recalling the definition of  $V_\Lambda^{(1)}$ , we get (cf. [2, Eqs. (4.12) and (4.15)])

$$\int_{[-K, K]^d} \tau_a(V_\Lambda^{(1)}) da \geq c\chi_{\Lambda_L} \quad \text{with } c > 0. \quad (7)$$

Combining (6), (7), and the lower bound on  $\|g\|$ , we get  $c(1 - \sqrt{E_0})^2 \leq c'E_0^{\frac{1}{4}}K^{d+1}$ , which leads to a contradiction for  $E_0 \approx \ell_0^{-(4(d+1)+)}$  and  $L_0$  large.

We may thus conclude that if  $\varrho$  is fixed,  $p > 0$  is given,  $E \in [0, E_0]$  with  $E_0 \approx (\varrho^{-1} \log L_0)^{-(\frac{4(d+1)}{d}+)}$ , and  $L_0$  is sufficiently large, then, with probability  $\geq 1 - L_0^{-p}$ , we have  $\|R_\Lambda(E)\| \lesssim E_0^{-1}$  and  $\|\chi_x R_\Lambda(E) \chi_y\| \lesssim e^{-cL_0}$  for  $x, y \in \Lambda$  with  $|x - y| \geq \frac{L_0}{10}$ . Moreover, it is clear that if  $V_\Lambda^{(2)} = \sum_{\zeta \in Y} u(x - \zeta)$ , the results are still valid if we replace  $V_\Lambda^{(2)}$  by  $\sum_{\zeta \in Y} t_\zeta u(x - \zeta)$  with arbitrary  $t_\zeta \in [0, 1]$ . We now declare all boxes  $\Lambda(j)$  with  $j \notin x_0 + 2\ell_0 \mathbb{Z}^d$  (and hence do not contribute to  $V_\Lambda^{(1)}$ ) to be *free boxes*. Moreover, inside the free boxes we use the representation of the Poisson process  $X$  by a thinned Poisson process (e.g., [10]), that is, by a Poisson process  $Y$  with density  $2\varrho$  in such a way that to each Poisson point  $\xi \in Y$  is attached a Bernoulli random variable  $\varepsilon_\xi$ ,  $\varepsilon_\xi = 0$  or  $1$  with equal probability, and the single-site contribution to the potential is given by  $\varepsilon_\xi u(x - \xi)$ . Note that any site  $\xi \in Y$  in a free box is a *free site* in the sense of [2].

The multiscale analysis now proceeds by induction. If  $\Lambda$  is a box of size  $L$ , we divide it into non-overlapping cubes  $\Lambda(w)$  of side  $\approx e^{-L^2}$  centered at points  $w \in e^{-L^2} \mathbb{Z}^d$ . With probability  $\geq 1 - L^{-p}$ ,  $p$

large, we require  $N(\Lambda) \lesssim \varrho L^d$  and all  $N(\Lambda(w)) \leq 1$ . We introduce an equivalence relation on Poisson configurations  $X_\Lambda$  in  $\Lambda$ ;  $\tilde{X}_\Lambda$  is the collection of Poisson configurations in  $\Lambda$  that cannot be distinguished from  $X_\Lambda$  by the counting functions  $N(\Lambda(w))$ . The crucial observation is that if we change a Poisson configuration to another one in the same equivalence class, then the eigenvalues of  $H_\Lambda$  in a fixed interval do not move by more than  $\lesssim e^{-L^2}$ . We may thus consider only the case when the Poisson points in  $\Lambda$  are in the lattice  $e^{-L^2} \mathbb{Z}^d$ , since the desired results will then hold for the whole equivalence class. This reduction allows the use of the results in [2], using equivalence classes of Poisson configurations instead of fixed Bernoulli configurations. Inside the free boxes equivalence classes are defined as above but for the Poisson points in  $Y$ . Since we have a finite number of equivalence classes of configurations inside a free box, we fix the points of the Poisson process  $Y$  in the free boxes, and conduct the analysis of [2, Lemmas 5.1], “tuning” the free parameters  $t_\xi$  to  $\varepsilon_\xi = 0$  or 1 to obtain “good” configurations, with a probability estimated by Sperner’s Lemma using [2, Lemma 3.1]. As in [2], we get the following result (cf. [2, Proposition A]), where  $\Lambda_L$  denotes a cube of side  $L$ .

**Proposition 1** *Given  $\varrho > 0$ , there exists  $E_0 = E_0(\varrho) > 0$  and  $L_0 = L_0(\varrho) < \infty$ , such that if  $\mathcal{X}_{\Lambda_L}(E)$  denotes the Poisson configurations for which*

$$\|R_{\Lambda_L}(E)\| \leq e^{L^{1-}} \quad \text{and} \quad \|\chi_x R_{\Lambda_L}(E) \chi_y\| \leq e^{-cL} \text{ for } |x - y| \geq \frac{L}{10}, \quad (8)$$

*then for all  $L \geq L_0$  and all  $E \in [0, E_0]$  we have*

$$\mathbb{P}_\varrho \{ \mathcal{X}_{\Lambda_L}(E) \} \geq 1 - \frac{1}{L^{\frac{3}{8}d-}}. \quad (9)$$

Proposition 1 provides a single-energy multiscale analysis. The weak probability estimate in (9) does not allow for an energy-interval multiscale analysis as in [4,7]. The first part of Theorem 1, namely exponential localization, requires the energy elimination scheme given in [2, Section 7]. To obtain the decay of the eigenfunction correlations given in (3) we add ideas from [8].

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